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by Mawardi Bahri Et Al

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Correlation Theorem for Two-Sided Quaternion Fourier Transform

Mawardi Bahri, Muh. Irwan, Syamsuddin Toaha and Muh. Saleh AF

Department of Mathematics, Hasanuddin University
Jl. Perintis Kemerdekaan KM 10 Tamalanrea Makassar 90245, Indonesia

Abstract

In this paper we establish correlation theorem for the two-sided quaternion Fourier transform (QFT). A consequence of the theorem is also investigated.

Keywords: quaternion Fourier transform, correlation

I. Introduction

The quaternion Fourier transform (QFT) is a nontrivial generalization of the classical Fourier transform (FT) using quaternion algebra. A number of already known and useful properties of this extended transform are generalizations of the corresponding properties of the FT with some modifications (see, for example, [1, 2, 3, 4, 5]). *One* of the *most* powerful *properties* of the QFT is the convolution theorem. Recently, in [6] authors proposed the convolution theorem for the two-sided QFT, which describes the relationship between the two-sided QFT and convolution of two quaternion function.

Therefore, the main objective of the present paper is to establish the correlation for the two-sided QFT, which is a generalization of correlation theorem of the classical FT. We find that the correlation theorem does not work well for the right-sided quaternion Fourier transform and left-sided quaternion Fourier transform.

The quaternion algebra over $\mathbb{R}$, denoted by

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3\}, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R}, \tag{1}$$

is an associative non-commutative four-dimensional algebra, which the quaternion units $\boldsymbol{i}, \boldsymbol{j},$ and $\boldsymbol{k}$ obey the following multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1,\ \boldsymbol{ij} = -\boldsymbol{ji},\ \boldsymbol{ik} = -\boldsymbol{ki},\ \boldsymbol{jk} = -\boldsymbol{kj}, \text{and } \boldsymbol{ijk} = -1. \tag{2}$$

The quaternion conjugate is defined by

$$\bar{q} = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3, \tag{3}$$

which is an anti-involution, that is,

$$\bar{\bar{p}} = p, \qquad \overline{p+q} = \bar{p} + \bar{q},\ \overline{pq} = \bar{q}\bar{p}. \tag{4}$$

The norm of a quaternion is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{{q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2}. \tag{5}$$

It is not difficult to check that

$$|qp| = |p||q|, \quad \forall\, p, q \in \mathbb{H}. \tag{6}$$

We further get the inverse

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

This fact shows that $\mathbb{H}$ is a skew field, that means, every nonzero element has a multiplicative inverse.

For the sake of further simplicity, we will use the real vector notations

$$\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2,\ f(\boldsymbol{x}) = f(x_1, x_2), f(\boldsymbol{\omega}) = f(\omega_1, \omega_2),$$

and so on when there is confusion.

2. Main Results

In this section we introduce the definition of the two-sided QFT and establish the correlation of two quaternion-valued functions associated with the two-sided QFT.

Definition 2.1 (Two-sided QFT) Let f be in $\in L^2(\mathbb{R}^2; \mathbb{H})$. *The two-sided QFT of the quaternion function f is the transform given by the integral*

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 x_1} f(\boldsymbol{x}) e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x}. \tag{7}$$

Theorem 2.1 (Inverse two-sided QFT) Suppose that $f \in L^2(\mathbb{R}^2; \mathbb{H})$ *and* $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$. *Then the two-sided QFT is an invertible transform and its inverse is given by*

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 x_1} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) e^{\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{\omega}. \tag{8}$$

Definition 2.2 (Quaternion Correlation) *Suppose* f *and* g *are quaternion functions in* $L^2(\mathbb{R}^2;\mathbb{H})$. *The quaternion correlation of the functions is given by*

$$(f\otimes g)(\boldsymbol{x})=\int_{\mathbb{R}^2} f(\boldsymbol{x}+\boldsymbol{y})\overline{g(\boldsymbol{y})}\,d^2\boldsymbol{y}. \tag{9}$$

The following theorem is the main result of this paper, which describes how the two-sided QFT behaves under correlations.

Theorem 2.2 *Let* $f,g\in L^2(\mathbb{R}^2;\mathbb{H})$ *be two quaternion-valued functions. The two-sided QFT of correlation of* f *and* g *takes the form*

$$\begin{aligned}
&\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})\\
&=\left(\mathcal{F}_q\{f_0\}(\boldsymbol{\omega})+\boldsymbol{i}\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\right)\left(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{j}\mathcal{F}\{g_2\}(-\omega_1,\omega_2)\right)\\
&\quad+\left(\mathcal{F}_q\{f_0\}(\omega_1,-\omega_2)+\boldsymbol{i}\mathcal{F}_q\{f_1\}(\omega_1,-\omega_2)\right)\left(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})-\boldsymbol{k}\mathcal{F}\{g_3\}(\omega_1,-\omega_2)\right)\\
&\quad+\left(\boldsymbol{j}\mathcal{F}_q\{f_2\}(-\omega_1,\omega_2)+\boldsymbol{k}\mathcal{F}_q\{f_3\}(-\omega_1,\omega_2)\right)\left(\mathcal{F}_q\{g_0\}(\omega_1,-\omega_2)-\boldsymbol{j}\mathcal{F}\{g_2\}(-\boldsymbol{\omega})\right)\\
&\quad+\left(\boldsymbol{j}\mathcal{F}_q\{f_2\}(-\boldsymbol{\omega})+\boldsymbol{k}\mathcal{F}_q\{f_3\}(-\boldsymbol{\omega})\right)\left(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(\omega_1,-\omega_2)-\boldsymbol{k}\mathcal{F}\{g_3\}(-\boldsymbol{\omega})\right).
\end{aligned}$$

Proof. Applying the two-sided QFT definition gives

$$\begin{aligned}
&\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})\\
&=\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1x_1}\left(\int_{\mathbb{R}^2} f(\boldsymbol{x}+\boldsymbol{y})\,\overline{g(\boldsymbol{y})}\,d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2x_2}\,d^2\boldsymbol{x}.
\end{aligned}$$

By inserting the change of variables $\boldsymbol{z}=\boldsymbol{x}+\boldsymbol{y}$ to the above expression we immediately obtain

$$\begin{aligned}
&\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})\\
&=\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(\int_{\mathbb{R}^2} f(\boldsymbol{z})\,\overline{g(\boldsymbol{y})}\,d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\,d^2\boldsymbol{z}\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(\left(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\right)+\left(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\right)\right)\\
&\qquad\left(\left(g_0(\boldsymbol{y})-\boldsymbol{j}g_2(\boldsymbol{y})\right)+\left(-\boldsymbol{i}g_1(\boldsymbol{y})-\boldsymbol{k}g_3(\boldsymbol{y})\right)\right)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\,d^2\boldsymbol{y}\,d^2\boldsymbol{z}\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\right)\left(g_0(\boldsymbol{y})-\boldsymbol{j}g_2(\boldsymbol{y})\right)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\,d^2\boldsymbol{y}\,d^2\boldsymbol{z}\\
&\quad+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\right)\left(-\boldsymbol{i}g_1(\boldsymbol{y})-\boldsymbol{k}g_3(\boldsymbol{y})\right)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\,d^2\boldsymbol{y}\,d^2\boldsymbol{z}\\
&\quad+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\right)\left(g_0(\boldsymbol{y})-\boldsymbol{j}g_2(\boldsymbol{y})\right)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\,d^2\boldsymbol{y}\,d^2\boldsymbol{z}
\end{aligned}$$

$$+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(\left(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\right)\left(-\boldsymbol{i}g_1(\boldsymbol{y})-\boldsymbol{k}g_3(\boldsymbol{y})\right)\right)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\; d^2\boldsymbol{y}\, d^2\boldsymbol{z}$$

$$= \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\left(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\right)$$

$$\left(\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\, g_0(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\; d^2\boldsymbol{y}-\boldsymbol{j}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1}g_2(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\; d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$\left(-\boldsymbol{i}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1)}g_1(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\; d^2\boldsymbol{y}-\boldsymbol{k}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1)}g_3(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\; d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\left(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\right)$$

$$\left(\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1)}\, g_0(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\; d^2\boldsymbol{y}-\boldsymbol{j}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}g_2(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\; d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\left(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\right)$$

$$\left(-\boldsymbol{i}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1)}g_1(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\; d^2\boldsymbol{y}-\boldsymbol{k}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1)}g_3(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\; d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$= \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\left(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\right)\left(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{j}\mathcal{F}_q\{g_2\}(\omega_1,-\omega_2)\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\left(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\right)\left(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})-\boldsymbol{k}\mathcal{F}_q\{g_3\}(\omega_1,-\omega_2)\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\left(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\right)\left(\mathcal{F}_q\{g_0\}(\omega_1,-\omega_2)-\boldsymbol{j}\mathcal{F}_q\{g_2\}(-\boldsymbol{\omega})\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\left(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\right)\left(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(\omega_1,-\omega_2)+\boldsymbol{k}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$= \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\left(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}\left(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{j}\mathcal{F}_q\{g_2\}(\omega_1,-\omega_2)\right)$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\big)\; e^{\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}\Big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})-\boldsymbol{k}\mathcal{F}_q\{g_3\}(\omega_1,-\omega_2)\Big)$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)\, e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}\Big(\mathcal{F}_q\{g_0\}(\omega_1,-\omega_2)-\boldsymbol{j}\mathcal{F}_q\{g_2\}(-\boldsymbol{\omega})\Big)$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)\; e^{\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}\,\big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(\omega_1,-\omega_2)-\boldsymbol{k}\mathcal{F}_q\{g_3\}(-\boldsymbol{\omega})\big)$$

$$\begin{aligned}
&=\Big(\mathcal{F}_q\{f_0\}(\boldsymbol{\omega})+\boldsymbol{i}\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\Big)\Big(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{j}\mathcal{F}\{g_2\}(-\omega_1,\omega_2)\Big)\\
&\quad+\Big(\mathcal{F}_q\{f_0\}(\omega_1,-\omega_2)+\boldsymbol{i}\mathcal{F}_q\{f_1\}(\omega_1,-\omega_2)\Big)\big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})-\boldsymbol{k}\mathcal{F}\{g_3\}(\omega_1,-\omega_2)\big)\\
&\quad+\Big(\boldsymbol{j}\mathcal{F}_q\{f_2\}(-\omega_1,\omega_2)+\boldsymbol{k}\mathcal{F}_q\{f_3\}(-\omega_1,\omega_2)\Big)\big(\mathcal{F}_q\{g_0\}(\omega_1,-\omega_2)-\boldsymbol{j}\mathcal{F}\{g_2\}(-\boldsymbol{\omega})\big)\\
&\quad+\Big(\boldsymbol{j}\mathcal{F}_q\{f_2\}(-\boldsymbol{\omega})+\boldsymbol{k}\mathcal{F}_q\{f_3\}(-\boldsymbol{\omega})\Big)\big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(\omega_1,-\omega_2)-\boldsymbol{k}\mathcal{F}\{g_3\}(-\boldsymbol{\omega})\big).
\end{aligned}$$

As a consequence of the above theorem, we immediately obtain

Lemma 2.3 *Given any two quaternion-valued functions* $f,g\in L^2(\mathbb{R}^2;\,\mathbb{H})$. *If we assume that* $\mathcal{F}_q\{f\}(\boldsymbol{\omega})\in L^2(\mathbb{R}^2;\,\mathbb{R})$, *then*

$$\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})=\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\Big(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{i}\,\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})\Big)$$

$$-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\Big(\boldsymbol{j}\mathcal{F}_q\{g_2\}(\omega_1,-\omega_2)+\boldsymbol{k}\mathcal{F}_q\{g_3\}(-\omega_1,\omega_2)\Big).$$

Proof. Straightforward computations show that

$$\begin{aligned}
&\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})\\
&=\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 x_1}\left(\int_{\mathbb{R}^2} f(\boldsymbol{x}+\boldsymbol{y})\overline{g(\boldsymbol{y})}\;d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 x_2}d^2\boldsymbol{x}\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\Big(f(\boldsymbol{z})\big(g_0(\boldsymbol{y})-\boldsymbol{i}g_1(\boldsymbol{y})\big)-f(\boldsymbol{z})\big(\boldsymbol{j}g_2(\boldsymbol{y})+\boldsymbol{k}g_3(\boldsymbol{y})\big)\Big)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}d^2\boldsymbol{y}\;d^2\boldsymbol{z}\\
&=\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}f(\boldsymbol{z})\int_{\mathbb{R}^2}\big(g_0(\boldsymbol{y})-\boldsymbol{i}g_1(\boldsymbol{y})\big)e^{\boldsymbol{j}\omega_2 y_2}d^2\boldsymbol{y}\;e^{-\boldsymbol{j}\omega_2 z_2}\;d^2\boldsymbol{z}\\
&\quad-\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}f(\boldsymbol{z})\int_{\mathbb{R}^2}(\boldsymbol{j}g_2(\boldsymbol{y})+\boldsymbol{k}g_3 y)e^{\boldsymbol{j}\omega_2 y_2}d^2\boldsymbol{y}\;e^{-\boldsymbol{j}\omega_2 z_2}\;d^2\boldsymbol{z}\\
&=\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1}\big(g_0(\boldsymbol{y})-\boldsymbol{i}g_1(\boldsymbol{y})\big)\;e^{-\boldsymbol{j}\omega_2 y_2}\;d^2\boldsymbol{y}
\end{aligned}$$

$$-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1}\left(\boldsymbol{j}g_2(\boldsymbol{y})+\boldsymbol{k}g_3(\boldsymbol{y})\right)e^{-\boldsymbol{j}\omega_2 y_2}\,d^2\boldsymbol{y}.$$

We further obtain

$$\begin{aligned}
&\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})\\
&\quad=\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\left(g_0(\boldsymbol{y})-\boldsymbol{i}g_1(\boldsymbol{y})\right)e^{\boldsymbol{j}\omega_2 y_2}\,d^2\boldsymbol{y}\\
&\quad\quad-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\left(\boldsymbol{j}g_2(\boldsymbol{y})+\boldsymbol{k}g_3(\boldsymbol{y})\right)e^{\boldsymbol{j}\omega_2 y_2}\,d^2\boldsymbol{y}\\
&\quad=\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\left(\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}g_0(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\,d^2\boldsymbol{y}-\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\boldsymbol{i}g_1(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\,d^2\boldsymbol{y}\right)\\
&\quad\quad-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\left(\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\boldsymbol{j}g_2(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\,d^2\boldsymbol{y}+\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\boldsymbol{k}g_3(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\,d^2\boldsymbol{y}\right)\\
&\quad=\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\left(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{i}\,\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})\right)\\
&\quad\quad-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\left(\boldsymbol{j}\mathcal{F}_q\{g_2\}(\omega_1,-\omega_2)+\boldsymbol{k}\mathcal{F}_q\{g_3\}(-\omega_1,\omega_2)\right).
\end{aligned}$$

which was to be proved.

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